MELLIN-BARNES INTEGRALS FOR STABLE DISTRIBUTIONS AND THEIR CONVOLUTIONS

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Abstract

In this expository paper we first survey the method of Mellin-Barnes integrals to represent the α stable Lévy distributions in probability theory ($0 < \alpha \leq 2$). These integrals are known to be useful for obtaining convergent and asymptotic series representations of the corresponding probability density functions. The novelty concerns the convolution between two stable probability densities of different Lévy index, which turns to be a probability law of physical interest, even if it is no longer stable and self-similar. A particular but interesting case of convolution is obtained combining the Cauchy-Lorentz density ($\alpha = 1$) with the Gaussian density ($\alpha = 2$) that yields the so-called Voigt profile. Our machinery can be applied to derive the fundamental solutions of space-fractional diffusion equations of two orders.

Introduction

Mellin-Barnes integrals are a family of integrals in the complex plane whose integrand is given by the ratio of products of Gamma functions. Despite of the name, the Mellin-Barnes integrals were initially introduced in 1888 by the Italian mathematician S. Pincherle [27] in a couple of papers on the duality principle between linear differential equations and linear difference equations with rational coefficients, as discussed by the Authors in [20]. Only at the beginning of the XX century they were widely adopted by Mellin and Barnes. The Mellin-Barnes integrals are strongly related with the Mellin transform, in particular with its inverse transformation, in the framework of the so-called Melling setting. As shown in [22], [29], see also [10], [16], [32], [36], a powerful method to evaluate integrals can be obtained based on the Mellin setting. The Mellin-Barnes integrals are also essential tools for treating higher transcendental functions as generalized hypergeometric functions $_pF_q$ and Meijer G- and Fox H-functions, see [17], [18], [23], [38]. Furthermore, they provide a useful representation to compute the asymptotic behaviour of functions, see [26], [40]. All the above machinery is also used in the theory of probability, see e.g. [21], [24], [34], [37], [39], [41], [42], and in theory of Fractional Calculus, see e.g. [4], [5], [6], [7], [8], [9], [15], [19], [28], [35]. We point out the forthcoming handbook on Mellin Transforms by Brychkov, Kilbas, Marichev and Prudnikov [3], that will surely become a table-text for most applied mathematicians.

In this paper, after the essential notions and notations concerning the Mellin-Barnes integrals and the Mellin transform, we recall how the α stable distributions in probability theory $(0 < \alpha \leq 2)$ can be represented by these tools. In the framework of *symmetric* distributions we illustrate our simple method to derive the known results. Then, we extend the method to derive the Mellin-Barnes representation for the probability densities obtained through a convolution of two stable densities of index α_1, α_2 $(0 < \alpha_1 < \alpha_2 \leq 2)$. The case $\{\alpha_1 = 1, \alpha_2 = 2\}$ is of physical interest since it is related to the so-called Voigt profile function, well-known in molecular spectroscopy and atmospheric radiative transfer. We show how the convolution is of general interest since it enters in the treatment of space-fractional diffusion equations of double order. Finally, we point out that our results, being based on simple manipulations, can be understood by non-specialists of transform methods and special functions; however they could be derived through a more general analysis involving the functions of the Fox type.

1. The Mellin setting: basic formulas

The Mellin-Barnes integrals are complex integrals which contain Gamma functions in their integrands as follows,

$$MB_{\pm}(z) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a_1 + A_1 s)\cdots\Gamma(a_m + A_m s)}{\Gamma(c_1 + C_1 s)\cdots\Gamma(c_p + C_p s)}$$

$$\times \frac{\Gamma(b_1 - B_1 s)\cdots\Gamma(b_n - B_n s)}{\Gamma(d_1 - D_1 s)\cdots\Gamma(d_q - D_q s)} \, z^{\pm s} \, ds \,. \tag{1.1}$$

It is assumed that γ is real, all the A_j, B_j, C_j, D_j are *positive*, and all the a_j, b_j, c_j, $_j$ are complex. The path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integrands. For more details see, e.g. [2], [11], [12], [26]. By using the residue theorem it is not difficult to formally expand these integrals in power series. The Melin-Barnes integrals are known to occur in the definitions of higher transcendental functions as generalized hypergeometric functions $_pF_q$ and Meijer G- and Fox H-functions, see [17], [18], [23], [38], being related to their Mellin transforms.

For convenience, let us here recall the essential formulas for this kind of integral transform, referring for details to specialized textbooks, e.g., [10], [16], [22], [3], [36]. Let

$$\mathcal{M}\{f(x); s\} = f^*(s) = \int_0^{+\infty} f(x)\, x^{s-1}\, dx, \quad \gamma_1 < \Re(s) < \gamma_2, \tag{1.2}$$

be the Mellin transform of a sufficiently well-behaved function $f(x)$, and let

$$\mathcal{M}^{-1}\{f^*(s); x\} = f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\, x^{-s}\, ds \tag{1.3}$$

be the inverse Mellin transform, where $x > 0$, $\gamma = \Re(s)$, $\gamma_1 < \gamma < \gamma_2$. It is straightforward to note that $MB_-(x)$ provide an essential tool for the inversion of Mellin transforms when these are expressed in terms products and ratios of Gamma functions.

Denoting by $\overset{\mathcal{M}}{\leftrightarrow}$ the juxtaposition of a function $f(x)$ with its Mellin transform $f^*(s)$, the main rules are:

$$f(ax) \overset{\mathcal{M}}{\leftrightarrow} a^{-s} f^*(s), \quad a > 0, \tag{1.4}$$

$$x^a f(x) \overset{\mathcal{M}}{\leftrightarrow} f^*(s+a), \tag{1.5}$$

$$f(x^p) \overset{\mathcal{M}}{\leftrightarrow} \frac{1}{|p|} f^*(s/p), \quad p \neq 0, \tag{1.6}$$

$$h_1(x) = \int_0^\infty \frac{1}{y} f(y) g(x/y) \, dy \overset{\mathcal{M}}{\leftrightarrow} h_1^*(s) = f^*(s) g^*(s), \tag{1.7}$$

$$h_2(x) = \int_0^\infty f(xy) g(y) \, dy \overset{\mathcal{M}}{\leftrightarrow} h_2^*(s) = f^*(s) g^*(1-s). \tag{1.8}$$

The most simple example of Mellin transform is provided by the Legendre integral representation of the Gamma function

$$\Gamma(s) = \int_0^\infty \mathrm{e}^{-x} x^{s-1} \, dx, \; \Re(s) > 0, \quad \text{so} \quad \mathrm{e}^{-x} \overset{\mathcal{M}}{\leftrightarrow} \Gamma(s). \tag{1.9}$$

Henceforth, the Mellin-Barnes integral representation of $\exp(-z)$ turns to be

$$\mathrm{e}^{-z} = \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) z^{-s} \, ds = \sum_{n=0}^\infty \frac{(-1)^n}{n!} z^n, \tag{1.10}$$

where $\mathcal{L}$ denotes a loop in the complex s plane which encircles the poles of $\Gamma(s)$ (in the positive sense) with endpoints at infinity in $\Re(s) < 0$ and with no restrictions on $\arg z$.

2. Stable distributions and their Mellin setting

The topic of stable distributions is a fascinating and fruitful area of research in probability theory; furthermore, nowadays, these distributions provide valuable models in physics, astronomy, finance, and communication theory. The general class of stable distributions was introduced and given this name by the French mathematician Paul Lévy in the early 1920's. Stable distributions have three "exclusive" properties, which can be briefly summarized stating that they 1) are "invariant under addition", 2) possess their own "domain of attraction", and 3) admit a "canonical characteristic function".

Referring to specialized textbooks for more details, see e.g. [1], [13], [31], [33], [39], [42], here we limit ourselves to consider the class of Lévy

strictly stable densities according to the Feller parameterization. Following Mainardi et al. [19], this class will be denoted by

$$\{L_\alpha^\theta(x)\}, \quad \text{with} \quad 0 < \alpha \le 2, \ |\theta| \le \min\{\alpha, 2 - \alpha\}, \ x \in \mathbf{R}, \qquad (2.1)$$

where α denotes the index of stability (or Lévy index) and θ is a real parameter related to the asymmetry, improperly referred to as the skewness. Then, the canonical characteristic function, namely the Fourier transform of the density (2.1), is

$$\mathcal{F}\{L_\alpha^\theta(x); \kappa\} = \widehat{L_\alpha^\theta}(\kappa) := \int_{-\infty}^{+\infty} e^{i\kappa x} L_\alpha^\theta(x)\, dx = \exp\left[-\psi_\alpha^\theta(\kappa)\right], \qquad (2.2)$$

where

$$\psi_\alpha^\theta(\kappa) := |\kappa|^\alpha\, e^{i(\operatorname{sign}\kappa)\theta\pi/2} = \overline{\psi_\alpha^\theta(-\kappa)} = \psi_\alpha^{-\theta}(-\kappa), \ \kappa \in \mathbf{R}. \qquad (2.3)$$

It is easy to note from their characteristic functions that the strictly stable densities are self-similar. Indeed, by setting with $a > 0$,

$$L_\alpha^\theta(x; a) \overset{\mathcal{F}}{\longleftrightarrow} \exp\left[-a|\kappa|^\alpha\, e^{i(\operatorname{sign}\kappa)\theta\pi/2}\right], \qquad (2.4)$$

we have

$$L_\alpha^\theta(x; a) = a^{-1/\alpha}\, L_\alpha^\theta\left(x/a^{1/\alpha}\right). \qquad (2.5)$$

For $\theta = 0$ we obtain *symmetric* densities of which noteworthy examples are provided by the Gaussian (or normal) law (with $\alpha = 2$) and the Cauchy-Lorentz law ($\alpha = 1$). The corresponding expressions are usually given as

$$p_G(x; \sigma) := \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/(2\sigma^2)} := L_2^0(x; a = \sigma^2/2), \quad x \in \mathbf{R}, \qquad (2.6)$$

where σ^2 denotes the variance, and

$$p_{CL}(x; \chi) := \frac{1}{\pi}\, \frac{\chi}{(x^2 + \chi^2)} := L_1^0(x; a = \chi), \quad x \in \mathbf{R}, \qquad (2.7)$$

where χ denotes the semi-interquartile range.

Convergent and asymptotic series expansions for stable densities were introduced by Feller [13] in the 1950's without be classified in the framework of a known class of special functions. A general representation of all stable distributions was only achieved in 1986 by Schneider [34], who, in his

remarkable (but almost ignored) paper, recognized, through the Mellin setting, that these distributions can be characterized in terms of H-functions. We point out hat only in the earliest 1960's this general class was been introduced into the realm of higher transcendental functions thanks to a pioneering article by Fox [14]. The result by Schneider can be summarized in his following expression of the Mellin transform for our $L_\alpha^\theta(x)$:

$$L_\alpha^{\theta\,*}(s) = \epsilon \, \frac{\Gamma(s)\,\Gamma(\epsilon - \epsilon s)}{\Gamma(\gamma - \gamma s)\,\Gamma(1 - \gamma + \gamma s)}\,, \quad \epsilon = \frac{1}{\alpha}\,, \quad \gamma = \frac{\alpha - \theta}{2\alpha}\,, \qquad (2.8)$$

valid in the strip $0 < \Re(s) < 1$. The inverse Mellin transform is thus given by

$$L_\alpha^\theta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L_\alpha^{\theta\,*}(s)\, x^{-s}\, ds\,, \qquad (2.9)$$

where the path of integration is the straight line from $c - i\infty$ to $c + i\infty$ with $0 < c < 1$. Then, the Mellin-Barnes representation turns out

$$L_\alpha^\theta(x) = \frac{\epsilon}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\,\Gamma(\epsilon - \epsilon s)}{\Gamma(\gamma - \gamma s)\,\Gamma(1 - \gamma + \gamma s)}\, x^{-s}\, ds\,. \qquad (2.10)$$

Schneider obtained Eq (2.8) by transforming the Fourier transform (2.2)-(2.3) into a Mellin transform through a suitable deformation of the integration path in the complex plane, and then, starting from his Mellin-Barnes representation, he interpreted the stable densities in terms of Fox H-functions.

In their 2001 paper on fundamental solutions of space-time fractional diffusion equations [19], Mainardi et al. obtained expressions of the Mellin-Barnes integrals for stable densities, see there Eq.(6.8)-(6.10), that are consistent with (2.10). We like to recall that stable densities evolving in time $L_\alpha^\theta(x;t)$ are solutions for the space-fractional diffusion equation

$$_xD_\theta^\alpha\, u(x,t) = \frac{\partial}{\partial t} u(x,t)\,, \quad x \in \mathbf{R}\,, \quad t \in \mathbf{R}^+\,, \qquad (2.11)$$

where the (real) field variable is subjected to the initial condition $u(x,0^+) = \delta(x)$. In (2.11) $_xD_\theta^\alpha$ denotes the so-called *Riesz-Feller* fractional derivative of order α ($0 < \alpha \leq 2$) and skewness θ ($|\theta| \leq \min\{\alpha, 2-\alpha\}$). This fractional derivative is just defined as the pseudo-differential operator whose symbol is the logarithm of the characteristic function of the stable density $L_\alpha^\theta(x)$. This means, for any sufficiently well-behaved function $f(x)$,

$$\mathcal{F}\{\,_xD_\theta^\alpha\, f(x); \kappa\} = -\psi_\alpha^\theta(\kappa)\, \widehat{f}(\kappa)\,, \qquad (2.12)$$

where $\psi_\alpha^\theta(\kappa)$ is given by (2.3). For $\alpha = 2$ ($\theta = 0$) this fractional derivative reduces to the standard derivative of order 2.

In [19] the analysis was more extended than that by Schneider since it was also devoted to other types of probability density functions as those generated by diffusion equations of fractional order in time. Furthermore, their method for deriving the Mellin-Barnes integrals was different from that of Schneider, even if equivalent, being based on the reduction of the relevant Fourier transform into a product of two Mellin transforms.

Here we like to offer a simpler and more direct method than those adopted in [19], [34], but, for sake of simplicity, limiting ourselves to the Mellin-Barnes representation for the *symmetric* stable densities. In this case we start from the inversion formula of Fourier transform

$$L_\alpha^0(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} e^{-|\kappa|^\alpha} d\kappa. \tag{2.13}$$

The Mellin-Barnes integral representation of (2.13) can be obtained starting from the Mellin-Barnes representation of the exponential function $e^{-i\kappa x}$, ($\kappa x \neq 0$), see (1.10):

$$e^{-i\kappa x} = \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) (i\kappa x)^{-s} ds, \tag{2.14}$$

where $\mathcal{L}$ is a loop in the complex s plane that encircles the poles of $\Gamma(s)$ in the positive sense, with end-points at infinity at $\Re(s) < 0$ and with no restriction on $\arg(i\kappa x)$.

Inserting (2.14) into (2.13) gives

$$\begin{aligned} L_\alpha^0(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) (i\kappa x)^{-s} ds \right\} e^{-|\kappa|^\alpha} d\kappa \\ &= \frac{1}{2\pi} \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \left\{ \int_{-\infty}^{+\infty} e^{-|\kappa|^\alpha} (i\kappa)^{-s} d\kappa \right\} x^{-s} ds, \end{aligned} \tag{2.15}$$

with the condition $\Re(s) < 1$ for convergence of term in brackets in the second line. Using the Legendre integral representation of the Gamma function (1.10) valid for $\Re(z) > 0$, the term in brackets can be re-written as follows

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-|\kappa|^\alpha} (i\kappa)^{-s} d\kappa &= \left[i^{-s} + (-i)^{-s} \right] \frac{1}{\alpha} \Gamma\left(\frac{1-s}{\alpha}\right) \\ &= 2 \sin\left(\frac{\pi}{2}(1+s)\right) \frac{1}{\alpha} \Gamma\left(\frac{1-s}{\alpha}\right) \\ &= \frac{2\pi}{\alpha} \frac{\Gamma\left(\frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(1 - \frac{1+s}{2}\right)}, \end{aligned} \tag{2.16}$$

which gives

$$L_\alpha^0(x) = \frac{1}{\alpha}\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(s)\Gamma\left(\frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\frac{1+s}{2}\right)}\, x^{-s}\, ds\,, \qquad (2.17)$$

with $0 < \Re(s) < 1$.

In view of the similarity property (2.5), the Mellin-Barnes integral representation for $L_\alpha^0(x; a)$ is easily derived from (2.17) as

$$L_\alpha^0(x; a) = \frac{1}{\alpha\, a^{1/\alpha}}\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(s)\,\Gamma\left(\frac{1-s}{\alpha}\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\frac{1+s}{2}\right)}\left(\frac{x}{a^{1/\alpha}}\right)^{-s}\, ds\,. \qquad (2.18)$$

3. Mellin setting for the generalized Voigt profile

Let us now consider the convolution of two symmetric stable densities of index α_1, α_2 ($0 < \alpha_1 < \alpha_2 \le 2$) with scale factors a_1, a_2, respectively:

$$\mathcal{V}\left(x; (\alpha_1, a_1), (\alpha_2, a_2)\right) = \int_{-\infty}^{+\infty} L_{\alpha_1}^0(x - \xi; a_1)\, L_{\alpha_2}^0(\xi; a_2)\, d\xi\,. \qquad (3.1)$$

This function provides the probability density function for the sum of two independent random variables obeying stable laws with different Lévy index. Being $\alpha_1 \ne \alpha_2$ the density (3.1) is no longer stable either self-similar. Furthermore, it can be interpreted as the fundamental solution of the space-fractional diffusion equation of distributed order

$$\left[c_1\, _xD_0^{\alpha_1} + c_2\, _xD_0^{\alpha_2}\right] u(x, t) = \frac{\partial}{\partial t}u(x, t)\,, \quad x \in \mathbf{R}\,, \quad t \in \mathbf{R}^+\,, \qquad (3.2)$$

with $a_1 = c_1 t$ and $a_2 = c_2 t$. We refer to our paper [24] for more details and instructive plots about this probability distribution and also on bibliography on the particular case (of physical relevance) $\{\alpha_1 = 1\,, \alpha_2 = 2\}$ known as *Voigt profile*. For the Voigt profile, see also the pre-print [25].

Here we limit ourselves to derive the Mellin-Barnes integral representation for the convoluted density (3.1), that we refer to as the *generalized Voigt profile*, by extending the method illustrated in the previous section for the single density, see (2.18). We start from the characteristic function (dropping in the L.H.S the dependence on Lévy indices and scale factors)

$$\widehat{\mathcal{V}}(\kappa) = e^{-a_1\,|\kappa|^{\alpha_1} - a_2\,|\kappa|^{\alpha_2}}\,, \qquad (3.3)$$

from which

$$\mathcal{V}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\kappa x} e^{-a_1 |\kappa|^{\alpha_1} - a_2 |\kappa|^{\alpha_2}} \, d\kappa = \frac{1}{2\pi} \left\{ Y_+(x) + Y_-(x) \right\}, \quad (3.4)$$

with

$$Y_{\pm}(x) = \int_0^{+\infty} e^{-(a_1 \kappa^{\alpha_1} \pm i\kappa x)} e^{-a_2 \kappa^{\alpha_2}} \, d\kappa. \quad (3.5)$$

Following our method of the previous section we use the Mellin-Barnes representation for the exponential functions $e^{\pm i\kappa x}$ and $e^{-a_1 k^{\alpha_1}}$, so the functions $Y_{\pm}(x)$ read

$$\begin{aligned}
Y_{\pm}(x) \;\; &= \int_0^{+\infty} e^{-a_2 \kappa^{\alpha_2}} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}_0} \right. \\
&\left. \frac{1}{2\pi i} \int_{\mathcal{L}_1} \Gamma(s_0)\, \Gamma(s_1) (\pm i\kappa x)^{-s_0} \, (a_1 \kappa^{\alpha_1})^{-s_1} \, ds_0 \, ds_1 \right\} d\kappa.
\end{aligned} \quad (3.6)$$

Because of the absolute convergence of this integral, we can interchange the order of integration and applying the definition of Gamma function we obtain

$$\begin{aligned}
Y_{\pm}(x) \;\; &= \frac{1}{\alpha_2} \frac{1}{2\pi i} \int_{\mathcal{L}_0} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \Gamma(s_0)\, \Gamma(s_1) \Gamma\left(\frac{1 - s_0 - \alpha_1 s_1}{\alpha_2} \right) \\
&\times (\pm ix)^{-s_0} \, a_2^{-(1 - s_0 - \alpha_1 s_1)/\alpha_2} \, a_1^{-s_1} \, ds_0 \, ds_1.
\end{aligned} \quad (3.7)$$

Now, considering that $\pm ix = x e^{\pm i\pi/2}$, the generalized Voigt function assumes the Mellin-Barnes integral representation

$$\begin{aligned}
\mathcal{V}(x) \;\; &= \frac{1}{\pi \alpha_2} \frac{1}{2\pi i} \int_{\mathcal{L}_0} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \Gamma(s_0)\, \Gamma(s_1)\, \Gamma\left(\frac{1 - s_0 - \alpha_1 s_1}{\alpha_2} \right) \\
&\times \cos(s_0 \,\pi/2)\, x^{-s_0} \, a_2^{-(1 - s_0 - \alpha_1 s_1)/\alpha_2} \, a_1^{-s_1} \, ds_0 \, ds_1.
\end{aligned} \quad (3.8)$$

The Mellin-Barnes integral representation of the special case of the Voigt profile ($\{\alpha_1 = 1, \, \alpha_2 = 2\}$) turns out to be expressed by a single integral in the complex plane. In this case the functions $Y_{\pm}(x)$ have the following Mellin-Barnes integral representations valid for $0 < \Re(s) < 1$

$$\begin{aligned}
Y_{\pm}(x) \;\; &= \int_0^{+\infty} e^{-a_2 \kappa^2} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)\, [(a_1 \pm ix)\kappa]^{-s} \, ds \right\} d\kappa \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \left\{ \int_0^{\infty} e^{-a_2 \kappa^2} \, \kappa^{-s} \, d\kappa \right\} (a_1 \pm ix)^{-s} \, ds, \\
&= \frac{1}{2 a^{1/2}} \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s) \Gamma\left(\frac{1}{2} - \frac{s}{2} \right) \left[\frac{(a_1 \pm ix)}{a_2^{1/2}} \right]^{-s} ds.
\end{aligned} \quad (3.9)$$

then the Voigt profile (denoted now by $V(x)$) turns out

$$
V(x) = \frac{1}{4\pi a_2^{1/2}} \left\{ \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)\,\Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \left[\frac{(a_1 + ix)}{a_2^{1/2}}\right]^{-s} ds \right.
$$
$$
\left. + \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \left[\frac{(a_1 - ix)}{a_2^{1/2}}\right]^{-s} ds \right\}. \tag{3.10}
$$

Finally, after lengthy manipulations involving H- and G- functions and detailed in [25], the Mellin-Barnes integral representation of the Voigt function (3.10) yields the following expression in terms of complementary error functions of complex conjugate arguments

$$
V(x) = \frac{1}{4\sqrt{\pi\, a_2}} \left\{ \mathrm{e}^{Z^2/4}\, \mathrm{erfc}(Z/2) + \mathrm{e}^{\overline{Z}^2/4}\, \mathrm{erfc}(\overline{Z}/2) \right\}, \tag{3.11}
$$

where, for convenience, we have set $Z = (a_1 + ix)/a_2^{1/2}$.

Conclusions

In this paper we have revisited the Mellin-Barnes integral representation for the stable distributions in probability theory, and we have illustrated a simple method to derive it. Such method has been extended to derive the Mellin-Barnes integrals for the convolution of two stable densities of different Lévy index, corresponding to a probability density that generalizes the Voigt profile function, well-known in molecular spectroscopy and atmospheric radiative transfer. The generalized Voigt density, when is evolving in time, can be interpreted as the fundamental solution of a fractional diffusion equation with two space derivatives of non-integer order.

Finally, our analysis can be considered as a first step of a machinery related to Mellin-Barnes integrals, the so-called Mellin setting, that can be pursued by researchers in analytical and numerical theory of special functions and fractional calculus.

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